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MINIMAL MONOMIAL REDUCTIONS AND THE REDUCED FIBER RING OF AN EXTREMAL IDEAL

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Abstract. Let I be a monomial ideal in a polynomial ring $A = K[x_1, \dots, x_n]$. We call a monomial ideal J to be a minimal monomial reduction ideal of I if there exists no proper monomial ideal $L \subset J$ such that L is a reduction ideal of I. We prove that there exists a unique minimal monomial reduction ideal J of I and we show that the maximum degree of a monomial generator of J determines the slope p of the linear function $\operatorname{reg}(I^t) = pt + c$ for $t \gg 0$. We determine the structure of the reduced fiber ring $\mathcal{F}(J)_{\mathrm{red}}$ of J and show that $\mathcal{F}(J)_{\mathrm{red}}$ is isomorphic to the inverse limit of an inverse system of semigroup rings determined by convex geometric properties of J.

Introduction

Let I be a monomial ideal in a polynomial ring $A = K[x_1, \dots, x_n]$ over a field K. Let $G(I)$ denote the unique minimal monomial set of generators of I.

Cutkosky-Herzog-Trung [5] and independently Kodiyalam [10] have shown that for any graded ideal I in a polynomial ring $A = K[x_1, \dots, x_n]$, the regularity of I^t is a linear function $pt+c$ for large enough t. Also the coefficient p of the linear function is known and it is given by the $\min\{\theta(J) : J$ is a graded reduction ideal of $I\}$, see [10]. Here $\theta(J)$ denotes the maximum of the degrees of elements in $G(J)$.

In Section 2 we give a convex geometric interpretation for this coefficient p for any monomial ideal $I \subset A$: let S be any set of monomials in A. We denote by $\Gamma(S) \subset \mathbb{N}^n$ the set of exponents of the monomials in S. Now let J be the monomial ideal which is determined by the property that $\Gamma(G(J)) = \operatorname{ext}(I)$, where $\operatorname{ext}(I)$ denotes the extreme points of the convex set $\operatorname{conv}(I)$. Here $\operatorname{conv}(I)$ denotes the convex hull of the elements of the set $\Gamma(I)$ in $\mathbb{R}^n$. This convex set is commonly called the *Newton polygon* of I. We show in Proposition 2.1 that the ideal J is the unique minimal monomial reduction ideal of I, that is, there exists no proper monomial ideal $L \subset J$ such that L is again a reduction ideal of I. It turns out that $p = \theta(J)$. In other words, $p = \max\{\deg x^a \colon a \in \operatorname{ext}(I)\}$.

We call a reduction ideal L of I to be a *Kodiyalam reduction* if $\theta(L) = p$. Thus the ideal J generated by monomials whose exponents belong to $\operatorname{ext}(I)$ is a Kodiyalam reduction.

We call a monomial ideal L to be an *extremal ideal* if $\Gamma(G(L)) = \operatorname{ext}(L)$. In other words, L is an extremal ideal if L is its own minimal monomial reduction. Notice that each squarefree monomial ideal is an extremal ideal. Let $\mu(L)$ denote the number of generators in a minimal generating set of a graded ideal L. It is easy to see that $\mu(\operatorname{Rad} I)$ is bounded above by $|\operatorname{ext}(I)|$ for any monomial ideal $I \subset A$.

In Section 3 we describe the faces of $\operatorname{conv}(I^m)$ for a monomial ideal I, and compare the supporting hyperplanes and the faces of $\operatorname{conv}(I^{n_1})$ and $\operatorname{conv}(I^{n_2})$ for two positive integers n_1, n_2.

In Section 4 we determine the structure of the reduced fiber ring $\mathcal{F}(L)_{\mathrm{red}}$ of an extremal ideal L. For any graded ideal $L \subset A = K[x_1, \ldots, x_n]$, the fiber ring $\mathcal{F}(L)$ is defined to be $\mathcal{R}(L)/\mathfrak{m}\mathcal{R}(L) = \bigoplus_{n\geq 0} L^n/mL^n$ where $\mathcal{R}(L)$ is the Rees ring and $\mathfrak{m} = (x_1, \ldots, x_n) \subset A$ is the graded maximal ideal of A. The main motivation to study the structure of the reduced fiber ring of an extremal ideal is to determine the dimension of the fiber ring of an arbitrary monomial ideal. Let $I \subset A$ be a monomial ideal and $J \subset I$ be its minimal monomial reduction. Then J is an extremal ideal, and $\dim \mathcal{F}(I) = \dim \mathcal{F}(J) = \dim \mathcal{F}(J)_{\mathrm{red}}$. So as far as dimension is concerned it is enough to consider the reduced fiber ring $\mathcal{F}(J)_{\mathrm{red}}$ of the extremal ideal J, whose structure is in general much simpler than that of $\mathcal{F}(J)$.

Let $\mathcal{F}_c$ denote the set of all compact faces of $\operatorname{conv}(I)$. It is shown in Lemma 3.1 that for each $F \in \mathcal{F}_c$, we have $F = \operatorname{conv}\{a_{j_1}, \ldots, a_{j_t}\}$ where $F \cap \operatorname{ext}(I) = \{a_{j_1}, \ldots, a_{j_t}\}$. For each $F \in \mathcal{F}_c$ we put $K[F] = K[x^{a_j}t \colon a_j \in F]$. As the main result of Section 4 we show in Theorem 4.9 that $\mathcal{F}(J)_{\mathrm{red}} \cong \varprojlim_{F\in\mathcal{F}_c} K[F]$. As an application to the Theorem 4.9 we get in the particular case of monomial ideals a result of Carles Bivia-Ausina [4] on the analytic spread of a Newton non-degenerate ideal.

Let $\overline{L}$ denotes the integral closure of an ideal L. In Section 5, using convex geometric arguments, we show in Theorem 5.1 that $\overline{I^\ell} = J\overline{I^{\ell-1}}$ where ℓ is the analytic spread of I. If we assume that I^a is integrally closed for $a \leq \ell - 1$, then as a corollary of Theorem 5.1, we obtain that $I^\ell = JI^{\ell-1}$, and that I is a normal ideal.

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1. Some Preliminaries on the Convex Geometry of Monomial Ideals

Let I be a monomial ideal in a polynomial ring $A = K[x_1, \ldots, x_n]$ over a field K. We denote by $G(I)$ the unique minimal monomial generating set of I.

For a monomial $u = x^a = x_1^{a(1)} \cdots x_n^{a(n)} \in A$ we denote by $\Gamma(u)$ the exponent vector $(a(1), \ldots, a(n))$ of u. Similarly, if S is any set of monomials in A, we set $\Gamma(S) = \{\Gamma(u) \colon u \in S\}$.

We denote the convex hull of $\Gamma(I)$ by $\operatorname{conv}(I)$. Here $\Gamma(I) = \{a : x^a \in I\}$. Recall that $\operatorname{conv}(I)$ is a polyhedron. A polyhedron can be defined as the intersection of finitely many closed half spaces. A polyhedron may also be thought of as the sum of a polytope (which is the convex hull of a finite set of points) and the positive cone generated by a finite set of vectors. Indeed these two notions are equivalent, (see [15, Theorem 1.2]).

Suppose that $G(I) = \{x^{a_1}, \ldots, x^{a_s}\}$, then

$$\operatorname{conv}(I) = \operatorname{conv}\{a_1, a_2, \ldots, a_s\} + \mathbb{R}^n_{\geq 0},$$

see [12, Lemma 4.3]. Here the positive cone $\mathbb{R}^n_{\geq 0}$ denotes the set of vectors $u \in \mathbb{R}^n$ such that $u(i) \geq 0$ for all $i = 1, \dots, n$. It follows that $\operatorname{conv}(I)$ is a polyhedron. It is called the *Newton polyhedron* of I.

Let $H_i = \{v \in \mathbb{R}^n \mid \langle v, u_i \rangle = c_i\}$ where $u_i \in \mathbb{R}^n$, $c_i \in \mathbb{R}$ for $i = 1, \dots, m$ be the hyperplanes in $\mathbb{R}^n$ such that $\operatorname{conv}(I) = \{v \in \mathbb{R}^n \mid \langle v, u_i \rangle \geq c_i,\ i = 1, \dots, m\}$. We observe

Lemma 1.1. *The vectors u_i belong to $\mathbb{R}^n_{\geq 0}$ for $i = 1, \dots, m$.*

Proof. We prove that $\langle e_j, u_i \rangle = u_i(j) \geq 0$ for all i, j. Here the vectors $e_j = (0, \dots, 0, 1, 0, \dots, 0)$ are the canonical unit vectors in $\mathbb{R}^n$ for $j = 1, \dots, n$ and 1 being at jth place. Let $a \in \Gamma(I)$, then $a + te_j \in \operatorname{conv}(I)$ for all j and $t \in \mathbb{R}_{\geq 0}$. Hence $\langle a + te_j, u_i \rangle \geq c_i$ for all i, j. Suppose $\langle e_{j_0}, u_{i_0} \rangle < 0$ for some j_0, i_0. Then we have $\langle a + te_{j_0}, u_{i_0} \rangle < c_{i_0}$ for $t \gg 0$, which is a contradiction. $\square$

Before proceeding further we need to set up some terminology from convex geometry (see [7]). We define the notions of exposed points and extreme points for a convex set $X \subset \mathbb{R}^n$. A point $a \in X$ is said to be an *extreme point*, provided all $b, c \in X$, $0 < \lambda < 1$, and $a = \lambda b + (1 - \lambda)c$ imply $a = b = c$. We denote this set of extreme points by $\operatorname{ext}(X)$.

Let $H = \{v \in \mathbb{R}^n \mid \langle v, u \rangle = c\}$ be a hyperplane where $u \in \mathbb{R}^n, c \in \mathbb{R}$. We denote by H_+ the nonnegative closed half space defined by H, i.e. $H_+ = \{v \in \mathbb{R}^n \mid \langle v, u \rangle \geq c\}$. We say H is a *supporting hyperplane* of a closed convex set X, if $X \subset H_+$ and $X \cap H \neq \emptyset$. Again, we may notice, as in Lemma 1.1 that for every supporting hyperplane $H = \{v \in \mathbb{R}^n \mid \langle v, u \rangle = c\} \subset \mathbb{R}^n$ of $\operatorname{conv}(I)$ one has $u \in \mathbb{R}^n_{\geq 0}$.

A set $F \subset X$ is called a *face* of X, if either $F = \emptyset$, or $F = X$, or if there exists a supporting hyperplane H of X such that $F = X \cap H$. We call F to be a *proper face* of X if $F \neq X$ and $F \neq \emptyset$.

Let F be a proper face of $\operatorname{conv}(I)$. Let $H = \{v \in \mathbb{R}^n \mid \langle v, u \rangle = c\}$ be a supporting hyperplane of $\operatorname{conv}(I)$ such that $F = H \cap \operatorname{conv}(I)$. It may be observed that F is a compact face of $\operatorname{conv}(I)$ if and only if the vector $u \in (\mathbb{R}_+ \backslash \{0\})^n$ i.e. $u(j) > 0$ for all $j = 1, \dots, n$.

We now define exposed points of X which we denote by $\exp(X)$. A point $a \in X$ is called an *exposed point* of X if the set $\{a\}$ consisting of single point is a face of X. Hence for every $a \in \exp(X)$ there exists a supporting hyperplane $H = \{v \in \mathbb{R}^n \mid \langle v, u \rangle = c\} \subset \mathbb{R}^n$ such that $\{a\} = X \cap H$ i.e. $\langle a, u \rangle = c$ and $\langle b, u \rangle > c$ for all $b \in X, b \neq a$.

We denote the extreme points of $\operatorname{conv}(I)$ by $\operatorname{ext}(I)$ and the exposed points of $\operatorname{conv}(I)$ by $\exp(I)$.

Proposition 1.2. *Let I be a monomial ideal in a polynomial ring $A = K[x_1, \dots, x_n]$ over a field K. Then, $a \in \exp(I)$ implies $x^a \in G(I)$.*

Proof. Let $a \in \exp(I)$ and $H = \{v \in \mathbb{R}^n \mid \langle v, u \rangle = c\} \subset \mathbb{R}^n$ be a supporting hyperplane of $\operatorname{conv}(I)$ such that $H \cap \operatorname{conv}(I) = \{a\}$. Notice that $u \in (\mathbb{R}_+/\{0\})^n$.

Let $G(I) = \{x^{a_1}, \dots, x^{a_s}\}$. Then $\operatorname{conv}(I) = \operatorname{conv}\{a_1, a_2, \dots, a_s\} + \mathbb{R}^n_{\geq 0}$. Therefore $a = \sum_{i=1}^s k_i a_i + v$ where $\sum_{i=1}^s k_i = 1$, $k_i \geq 0$, $v \in \mathbb{R}^n_{\geq 0}$. Now, since $\langle a_i, u \rangle \geq c$ and $\langle w, u \rangle > 0$ for any $0 \neq w \in \mathbb{R}^n_{\geq 0}$, $\langle a, u \rangle = c$ implies $a = a_i$ for some i. $\square$

Remark 1.3. For any closed convex set $X \subset \mathbb{R}^n$, one has $\exp(X) \subset \operatorname{ext}(X)$ and $\operatorname{ext}(X) \subset \operatorname{cl}(\exp(X))$ where $\operatorname{cl}(\exp(X))$ denotes the closure of X in $\mathbb{R}^n$ with respect to usual topology (see [7, Statement 3 and 9, Section 2.4]). In case $X = \operatorname{conv}(I)$, one has $\exp(I)$ is a finite set. Therefore $\operatorname{cl}(\exp(I)) = \exp(I)$, and hence $\exp(I) = \operatorname{ext}(I) \subset \Gamma(G(I))$.

2. Minimal Monomial Reduction Ideal

In this section we show that for any monomial ideal $I \in A = K[x_1, \ldots, x_n]$, there exists a unique minimal monomial reduction ideal J of I. We also show that the minimal monomial reduction ideal J of a monomial ideal I is a Kodiyalam reduction of I

Let $L \subset A = K[x_1, \ldots, x_n]$ be a graded ideal. An ideal $N \subset L$ is said to be a *reduction ideal of* L, if there exists a positive integer m such that $NL^{m-1} = L^m$. Let $\bar{I}$ denote the integral closure of an ideal I. It is known that $N \subset L$ is a reduction ideal of L if and only if $\overline{N} = \overline{L}$(see [3, Exercise 10.2.10(c)]).

Now let $I \subset K[x_1, \ldots, x_n]$ be a monomial ideal. We say a monomial ideal $J \subset I$ a *minimal monomial reduction ideal of* I if there exists no proper monomial ideal $J' \subset J$ such that J' is a reduction ideal of I. For a monomial ideal one has

$$\Gamma(\bar{I}) = \operatorname{conv}(I) \cap \mathbb{N}^n$$

(see [6, Exercise 4.22]). Hence a monomial ideal $J \subset I$ is a reduction ideal of I if and only if $\operatorname{conv}(J) = \operatorname{conv}(I)$.

Proposition 2.1. *Let I be a monomial ideal in a polynomial ring $A = K[x_1, \ldots, x_n]$ over a field K with $\operatorname{ext}(I) = \{a_1, \ldots, a_r\}$. Then the ideal $J = (x^{a_1}, \ldots, x^{a_r})$ is the unique minimal monomial reduction ideal of I.*

Proof. To show that J is a reduction ideal of I is equivalent to prove that $\operatorname{conv}(I) = \operatorname{conv}(J)$. For any monomial ideal $L \subset A$, we know that $\operatorname{conv}(L) = \operatorname{conv}(\Gamma(G(L))) + \mathbb{R}^n_{\geq 0}$, [12, Lemma 4.3]. We also have $\Gamma(G(J)) = \operatorname{ext}(I)$. On the other hand it follows easily from [13, Section 8.9] that

$$\operatorname{conv}(I) = \operatorname{conv}(\operatorname{ext}(I)) + \mathbb{R}^n_{\geq 0}. \tag{1}$$

These facts imply that $\operatorname{conv}(I) = \operatorname{conv}(J)$.

Again, it is also easy to see that J is the unique minimal monomial reduction of I. In fact, let L be any other monomial reduction ideal of I. We show that $J \subset L$. We have $\operatorname{conv}(I) = \operatorname{conv}(L)$, and so $\operatorname{ext}(L) = \operatorname{ext}(I)$. We know that $\operatorname{ext}(L) = \exp(L) \subset \Gamma(G(L))$, by Lemma 1.2. Therefore we have $\Gamma(G(J)) \subset \Gamma(G(L))$. $\square$

For all nonnegative integers m, we define the ideal $J^{[m]} := (x^{ma_1}, \ldots, x^{ma_r})$.

Corollary 2.2. *The ideal $J^{[m]}$ is the unique minimal monomial reduction ideal of I^m for all m.*

Proof. Let us fix an m, and denote by J_m the unique monomial reduction ideal of I^m. First notice that $J^{[m]}$ is a monomial reduction ideal of I^m. Indeed, as $J^{[m]}$ is

a monomial reduction ideal of J^m and J^m is a monomial reduction ideal of I^m, we have $J^{[m]}$ is a reduction ideal of I^m. Therefore $J_m \subset J^{[m]}$, by Theorem 2.1.

Next we claim that $\text{ext}(I^m) \supset \{ma_1, \ldots, ma_r\}$, and this will imply that $J^{[m]} \subset J_m$, by Theorem 2.1.

Let $H_i = \{v \in \mathbb{R}^n \mid \langle v, u_i \rangle = c_i\}$ be a supporting hyperplane of $\text{conv}(I)$ such that $H_i \cap \text{conv}(I) = \{a_i\}$ for $i = 1, \ldots, r$. We define the hyperplanes $mH_i = \{v \in \mathbb{R}^n \mid \langle v, u_i \rangle = mc_i\}$, $i = 1, \ldots, r$ and show that mH_i is a supporting hyperplane of $\text{conv}(I^m)$ with $mH_i \cap \text{conv}(I^m) = \{ma_i\}$. This then will imply the above claim.

It is clear that $ma_i \in mH_i \cap \text{conv}(I^m)$. Now let $a \in \Gamma(I^m)$ be an arbitrary element. Then $a = \sum_{j=1}^m a_{i_j} + v$ where $v \in \mathbb{N}^n$. It follows that $\langle a, u_i \rangle \geq mc_i$, and is equal to mc_i if and only if $a = ma_i$, as $u_i \in (\mathbb{R}_+/\{0\})^n$. It follows that $\langle b, u_i \rangle \geq mc_i$ for all $b \in \text{conv}(I^m)$, and equality holds if and only if $b = ma_i$. $\square$

Let I be a graded ideal in a polynomial ring $A = K[x_1, \ldots, x_n]$ over a field K. The ith regularity of an ideal I is defined to be $\text{reg}_i(I) = \max\{j : \text{Tor}_i^A(I, K)_{i+j} \neq 0\}$ and the Castelnuovo-Mumford regularity of I is defined to be $\text{reg}(I) = \max\{\text{reg}_i(I) - i\}$.

Cutkosky-Herzog-Trung [5] and independently Kodiyalam [10] have shown that $\text{reg}(I^t) = pt + c$ for $t \gg 0$. Also the coefficient of the linear function is known and it is given by

$$p = \min\{\theta(J) : J \text{ is a graded reduction ideal of } I\},$$

see [10]. Here $\theta(J)$ denotes the maximum of the degrees of elements in $G(J)$. We define a reduction ideal J of I to be a *Kodiyalam reduction* if $\theta(J) = p$.

More generally, it is shown in [5] that $\text{reg}_i(I^t) = p_i t + q_i$ for $t \gg 0$ are linear functions. From the arguments in Kodiyalam's paper [10] it follows immediately that $p_0 = p$.

Corollary 2.3. *Let I be a monomial ideal in $K[x_1, \ldots, x_n]$, then the minimal monomial reduction ideal J of I is a Kodiyalam reduction.*

Proof. The proof is very much on the line of arguments of Kodiyalam (see [10, Proposition 4]). By the very definition of p, we have $\theta(J) \geq p$. We now show that $\theta(J) \leq p$. It is enough to find a monomial reduction ideal L such that $\theta(L) \leq p$, as $G(J) \subset G(L)$ because $\Gamma(G(J)) = \text{ext}(I) = \text{ext}(L) \subset \Gamma(G(L))$. Notice that $\text{ext}(I) = \text{ext}(L)$, as $L \subset I$ being a reduction ideal of I, we have $\text{conv}(I) = \text{conv}(L)$.

Consider the minimal monomial generating system of I, given by $f_1, \ldots, f_s$ where $\deg f_i = d_i$ for all i and $d_1 \leq \cdots \leq d_s$. Let j be the largest integer such that $f_j^k \notin \mathfrak{m} I^k$ for any k where $\mathfrak{m}$ is the maximal graded ideal in A. Then $\text{reg}_0(I^t) \geq d_j t$ for all t. Set $L = (f_1, \ldots, f_j)$ and $P = (f_{j+1}, \ldots, f_s)$. Clearly L is a monomial ideal with $\theta(L) = d_j$. We claim that L is a reduction ideal of I. By the very choice of j, $P^t \subset \mathfrak{m} I^t$ for some t. Then $I^t = (L + P)^t = L(L + P)^{t-1} + P^t \subset LI^t + \mathfrak{m} I^t$. Hence by Nakayama's lemma, it follows that L is a reduction ideal of I. Now as $\theta(L) = d_j$ and $d_j t \leq pt + q_0$ for $t \gg 0$. We have $d_j \leq p$. Hence $\theta(L) \leq p$. $\square$

We call a monomial ideal L an *extremal ideal*, if $G(L) = \text{ext}(L)$. In other words, a monomial ideal L is an extremal ideal if it is the minimal monomial reduction of itself. In particular, the ideal J in Theorem 2.1 is an extremal ideal.

Remarks 2.4. 1. Every squarefree monomial ideal is an extremal ideal. Let $N \subset A$ be a squarefree monomial ideal and let $x^a \in G(N)$ be a monomial generator. We show that $a \in \text{ext}(N)$. As N is squarefree, for all i, one has $a(i) = 1$ or $a(i) = 0$. Let $r \leq n$ be the cardinality of i's such that $a_i = 1$. We define a vector $u \in \mathbb{N}^n$ given by $u(i) = 1$ if $a(i) = 1$ and $u(i) = n + 1$ if $a(i) = 0$. We claim that the hyperplane $S = \{v \in \mathbb{R}^n : \langle v, u\rangle = r\}$ is a supporting hyperplane of $\text{conv}(N)$ with $S \cap \text{conv}(N) = \{a\}$, which will imply that $a \in \text{ext}(N)$. Clearly, $S \cap \text{conv}(N) \supset \{a\}$. Let $b \in \text{conv}(N) = \text{conv}(\Gamma(G(N)) + \mathbb{R}^n_{\geq 0}$ with $b \neq a$ be an arbitrary element. We claim that $\langle b, u\rangle > r$. Notice that it is enough to consider $b \in \Gamma(G(N))$. Since $x^a, x^b \in G(N)$, we notice that there exists an i such that $b(i) = 1$ and $a(i) = 0$. Hence $\langle b, u\rangle \geq n + 1$ and so $\langle b, u\rangle > r$. Hence the claim.

Let $\mu(L)$ denote the number of generators in a minimal generating set of a graded ideal L.

2. Let $I \subset A$ be a monomial ideal. Then we have $\mu(\text{Rad}\, I) \leq |\,\text{ext}(I)|$. Infact, let $J \subset I$ be the minimal monomial reduction ideal of I. Then one has $\text{Rad}\, J = \text{Rad}\, I$. Hence $\mu(\text{Rad}\, I) = \mu(\text{Rad}\, J) \leq \mu(J) = |G(J)| = |\,\text{ext}(I)|$.

3. A description of the faces of $\text{conv}(I^m)$

Let $I = (x^{a_1}, x^{a_2}, \ldots, x^{a_s}) \subset A = K[x_1, \ldots, x_n]$ be a monomial ideal. We may assume that $\text{ext}(I) := \{a_1, \ldots, a_r\}$ is the set of extremal points of the convex hull of I after a proper rearrangement of generators. Then $J = (x^{a_1}, x^{a_2}, \ldots, x^{a_r})$ is the minimal monomial reduction ideal of I, see Theorem 2.1.

Next we consider the set of faces of $\text{conv}(I)$. Let $\mathcal{F}$ denote the set of proper faces and $\mathcal{F}_c \subset \mathcal{F}$ denote the set of compact faces of $\text{conv}(I)$. Let $F \in \mathcal{F}$ and $S := \{v \in \mathbb{R}^n \mid \langle v, u\rangle = c\}$ be a supporting hyperplane of $\text{conv}(I)$ such that $S \cap \text{conv}(I) = F$. It may be observed that $F \in \mathcal{F}_c$ if and only if the vector $u \in (\mathbb{R}_+ \backslash \{0\})^n$. For $j = 1, \ldots, n$, we define $e_j = (0, \ldots, 0, 1, \ldots, 0) \in \mathbb{R}^n$ to be the unit vectors, 1 being at jth place. With this notation, we have

Lemma 3.1. *Let $F \in \mathcal{F}$ be a face of $\text{conv}(I)$, and let $S = \{v \in \mathbb{R}^n : \langle v, u\rangle = c\}$ be a supporting hyperplane of $\text{conv}(I)$ such that $F = S \cap \text{conv}(I)$. Then $F \cap \text{ext}(I) \neq \emptyset$, and*

$$F = \text{conv}\{a_{j_1}, \ldots, a_{j_t}\} + \sum_{\{j:\, u(j)=0\}} \mathbb{R}_{\geq 0} e_j,$$

where $F \cap \text{ext}(I) = \{a_{j_1}, \ldots, a_{j_t}\}$.

Proof. Let $a \in \text{conv}(I)$. Then $a = \sum_i^r k_i a_i + v$ with $\sum k_i = 1$, $k_i \geq 0$, $v \in \mathbb{R}^n_{\geq 0}$ by Equation 1. Suppose $F \cap \text{ext}(I) = \emptyset$. Then $\langle a_i, u\rangle > c$ for all $i = 1, \ldots, r$. Therefore we have $\langle a, u\rangle > c$. Hence $F = S \cap \text{conv}(I) = \emptyset$, a contradiction.

Now let $F \cap \text{ext}(I) = \{a_{j_1}, \ldots, a_{j_t}\}$. First let F be a compact face, then $u \in (\mathbb{R}_+ \backslash \{0\})^n$. As $\langle a_i, u\rangle > c$ for all $a_i \in \text{ext}(I) \backslash \{a_{j_1}, \ldots, a_{j_t}\}$ and $\langle v, u\rangle > 0$ for all $0 \neq v \in \mathbb{R}^n_{\geq 0}$, we notice that $\langle a, u\rangle = c$ if and only if $a \in \text{conv}\{a_{j_1}, \ldots, a_{j_t}\}$. Hence $F = \text{conv}\{a_{j_1}, \ldots, a_{j_t}\}$.

Now let F be an noncompact face and let $Z = \{j : u(j) = 0\}$. Notice that the set $Z \neq \emptyset$. As $\langle a_i, u\rangle > c$ for all $a_i \in \text{ext}(I) \backslash \{a_{j_1}, \ldots, a_{j_t}\}$ and $\langle v, u\rangle \geq 0$ for all $v \in \mathbb{R}^n_{\geq 0}$

with $\langle v,u\rangle = 0$ if and only if $v \in \sum_{j\in Z} \mathbb{R}_{\geq 0} e_j$, we see that $\langle a,u\rangle = c$ if and only if $a \in \operatorname{conv}\{a_{j_1},\dots,a_{j_t}\} + \sum_{\{j\colon u(j)=0\}} \mathbb{R}_{\geq 0} e_j$. □

As an immediate consequence of Lemma 3.1 we obtain

Corollary 3.2. *Let $S = \{v \in \mathbb{R}^n \mid \langle v,u\rangle = c\}$ be a hyperplane. Then S is a supporting hyperplane of $\operatorname{conv}(I)$ if and only if $\langle a_i,u\rangle \geq c$ for all $a_i \in \operatorname{ext}(I)$ and $\langle a_j,u\rangle = c$ for some $a_j \in \operatorname{ext}(I)$.*

Lemma 3.3. *Let $S = \{v \in \mathbb{R}^n \mid \langle v,u\rangle = c\}$ where $u \in \mathbb{R}^n$, $c \in \mathbb{R}$, be a hyperplane, and let $n_1, n_2 \geq 1$ two integers and $q = n_2/n_1$. Then S is a supporting hyperplane of $\operatorname{conv}(I^{n_1})$ if and only if $qS = \{v \in \mathbb{R}^n \mid \langle v,u\rangle = qc\}$ is supporting hyperplane of $\operatorname{conv}(I^{n_2})$.*

Proof. We know by Corollary 2.2 that $\operatorname{ext}(I^m) = (ma_1,\dots,ma_r)$ for all $m \geq 1$. Now S is a supporting hyperplane of $\operatorname{conv}(I^{n_1})$ if and only if $\langle n_1a_i,u\rangle \geq c$ for all $n_1a_i \in \operatorname{ext}(I^{n_1})$ and $\langle n_1a_j,u\rangle = c$ for some $n_1a_j \in \operatorname{ext}(I^{n_1})$. This is the case if and only if $\langle n_2a_i,u\rangle = \langle (n_2/n_1)n_1a_i,u\rangle = q\langle n_1a_i,u\rangle \geq qc$ and $\langle n_2a_j,u\rangle = q\langle n_1a_j,u\rangle = qc$. This is equivalent to say that qS is a supporting hyperplane of $\operatorname{conv}(I^{n_2})$, see Corollary 3.2. □

Let $\mathcal{F}$ be the set of proper faces of $\operatorname{conv}(I)$. For each $F \in \mathcal{F}$ we choose a hyperplane $S = \{v \in \mathbb{R}^n \mid \langle v,u\rangle = c\}$ with $F = S \cap \operatorname{conv}(I)$. Then by Lemma 3.3, for any nonnegative integer m, the hyperplane mS is a supporting hyperplane of $\operatorname{conv}(I^m)$, and we set $mF = mS \cap \operatorname{conv}(I^m)$. It is easy to see that this definition does not depend on the choice of S. Indeed,

$$mF = \operatorname{conv}\{ma_{j_1},\dots,ma_{j_t}\} + \sum_{\{j\colon u(j)=0\}} \mathbb{R}_{\geq 0} e_j$$

if $F \cap \operatorname{ext}(I) = \{a_{j_1},\dots,a_{j_t}\}$. We denote by $m\mathcal{F}$ the set of proper faces of $\operatorname{conv}(I^m)$. As an immediate consequence of Lemma 3.3 we get

Corollary 3.4. *The map $\mathcal{F} \to m\mathcal{F}$, $F \mapsto mF$ is bijective.*

4. The structure of the reduced fiber ring of an extremal ideal

The main result of this section is Theorem 4.9 which gives us the structure of the reduced fiber ring of an extremal ideal. We proceed gradually towards it preparing the ground to prove it. We will use all the notation from previous section.

Recall that a monomial ideal $L \subset A = K[x_1,\dots,x_n]$ is said to be an extremal ideal if $\Gamma(G(L)) = \operatorname{ext}(L)$. In other words an extremal ideal is the minimal monomial reduction of itself, see Proposition 2.1.

The main motivation to study the structure of the reduced fiber ring $\mathcal{F}(J)_{\text{red}}$ of an extremal ideal is to determine the dimension of the fiber ring $\mathcal{F}(I)$ for any monomial ideal I. As one notices that $\dim \mathcal{F}(I) = \dim \mathcal{F}(J) = \dim \mathcal{F}(J)_{\text{red}}$, therefore it is enough to consider the reduced fiber ring $\mathcal{F}(J)_{\text{red}}$ as far as the dimension is concerned. We will see that in general the structure of the reduced fiber ring of an extremal ideal is more simple than that of the original fiber ring.

For the proof of Theorem 4.3 we shall need the following

Lemma 4.1. *Let $a = \sum_{i=1}^{r} l_i a_i$ where l_i are nonnegative integers, $\sum l_i = m$ and $\operatorname{ext}(I) = \{a_1, \ldots, a_r\}$. If $\{a_i : l_i \neq 0\} \not\subset F$ for some $F \in \mathcal{F}$, then $a \notin mF$.*

Proof. Let $S = \{v \in \mathbb{R}^n \mid \langle v, u\rangle = c\}$ be a supporting hyperplane of $\operatorname{conv}(I)$ such that $S \cap \operatorname{conv}(I) = F$. Then $mS = \{v \in \mathbb{R}^n \mid \langle v, u\rangle = mc\}$ is a supporting hyperplane of $\operatorname{conv}(I^m)$ such that $mS \cap \operatorname{conv}(I^m) = mF$.

Suppose that $a \in mF$. Then we have $\langle a, u\rangle = mc$. Since $\{a_i : l_i \neq 0\} \not\subset F$, there exists at least one j such that $\langle a_j, u\rangle > c$ which implies $\langle a, u\rangle > mc$, a contradiction. $\square$

Remark 4.2. From the above lemma, it follows that if the set $\{a_i : l_i \neq 0\} \not\subset F$ for any $F \in \mathcal{F}$, then $a \notin G$ for any $G \in m\mathcal{F}$. Indeed, as for every $G \in m\mathcal{F}$ there exists $F \in \mathcal{F}$ such that $G = mF$, by Corollary 3.4.

The following theorem is crucial in our study of the structure of the reduced fiber ring of an extremal ideal.

Theorem 4.3. *Let J be an extremal ideal with $G(J) = \{f_1, \ldots, f_r\}$ and $f_j = x^{a_j}$ for $j = 1, \ldots, r$. Let $Z = \{a_{j_1}, \ldots, a_{j_t}\}$ be a subset of $\Gamma(G(J))$. Then the following conditions are equivalent:*

(1) *$Z \subset F$ for some compact face $F \in \mathcal{F}$;*
(2) *For all $l_i \geq 0$ one has $f_{j_1}^{l_1} \cdots f_{j_t}^{l_t} \in G(J^m)$ where $m = \sum_{i=1}^{t} l_i$;*
(3) *For all $l_i \gg 0$ one has $f_{j_1}^{l_1} \cdots f_{j_t}^{l_t} \in G(J^m)$ where $m = \sum_{i=1}^{t} l_i$.*

Proof. (1) $\Rightarrow$ (2) Suppose there exists some nonnegative integers l_i such that $f' = f_{j_1}^{l_1} \cdots f_{j_t}^{l_t} \notin G(J^m)$ where $m = \sum l_i$. Then there exists $g \in G(J^m)$ such that $f' = hg$ where $\deg h > 0$. Let $S := \{v \in \mathbb{R}^n \mid \langle v, u\rangle = c\}$ be a supporting hyperplane such that $F = S \cap \operatorname{conv}(J)$. Notice that as F is a compact face, the vector u belongs to $(\mathbb{R}_+ \backslash \{0\})^n$. Now since the set $Z \subset F$, $\langle a_{j_k}, u\rangle = c$ for all $k = 1, \ldots, t$. Then we have $\langle \Gamma(f'), u\rangle = mc$, but since $\langle \Gamma(h), u\rangle > 0$ and $\langle \Gamma(g), u\rangle \geq mc$, one has $\langle \Gamma(hg), u\rangle > mc$, a contradiction.

(2) $\Rightarrow$ (3) is trivial.

(3) $\Rightarrow$ (1) Suppose if $Z \not\subset F$ for any compact face $F \in \mathcal{F}$, then we prove that for all $l_i \gg 0$ we have $f_{j_1}^{l_1} \cdots f_{j_t}^{l_t} \notin G(J^m)$ where $m = \sum_{i=1}^{t} l_i$.

Let $f = f_{j_1} \cdots f_{j_t}$. We will show that $f^{m_0} = f_{j_1}^{m_0} \cdots f_{j_t}^{m_0} \notin G(J^{m_0 t})$ for some positive integer m_0. From which it clearly follows that $f_{j_1}^{l_1} \cdots f_{j_t}^{l_t} \notin G(J^m)$ for all $l_i \geq m_0$ where $m = \sum l_i$.

Notice that in order to show that $f^m \notin G(J^{mt})$ for some m, it is enough to show that $f^k \notin G(\overline{J^{kt}})$ for some k. As let $f^k \notin G(\overline{J^{kt}})$ for some k. Then, $f^k = gh$ where $h \in G(\overline{J^{kt}})$ and $\deg g > 0$. Now as $h \in G(\overline{J^{kt}})$, $h^{k_1} \in J^{ktk_1}$ for some k_1 which implies $f^{kk_1} = g^{k_1} h^{k_1} \notin G(J^{ktk_1})$. Hence taking $m = kk_1$, we have $f^m \notin G(J^{mt})$.

We have assumed that $Z \not\subset F$ for any compact face $F \in \mathcal{F}$, but nevertheless Z may be a subset of a noncompact face in $\mathcal{F}$. We divide the proof in two cases depending on whether Z is a subset of some noncompact face or not.

Case 1: First we assume that $Z \not\subset F$ for any face (compact or noncompact) $F \in \mathcal{F}$. Suppose $f^m \in G(\overline{J^{mt}})$ for all m. Without loss of generality, let $x_1 | f$. Since

$f \in G(\overline{J^t})$, $g = f/x_1 \notin \overline{J^t}$. Hence $f \in \operatorname{conv}(J^t)$ and $g \notin \operatorname{conv}(J^t)$. Let l be the line segment joining $\Gamma(f)$ and $\Gamma(g)$. Then l intersects $\operatorname{conv}(J^t)$ at some point $p \in tF$ where F is a face of $\operatorname{conv}(J)$, see Corollary 3.4. Notice that $p \neq \Gamma(f)$, see Remark 4.2. Hence, $\Gamma(f) = p + v$ where $0 < \|v\| < 1$. Now for any m, consider the line segment joining $\Gamma(f^m)$ and $\Gamma(g^m)$, we denote this line segment by ml. We have $\Gamma(f^m) = mp + mv$ where $mp \in mtF$ and mtF is a face of $\operatorname{conv}(J^{mt})$. Again as $f^m \in G(\overline{J^{mt}})$, $f^m/x_1 \notin \overline{J^{mt}}$. Notice that $\Gamma(f^m/x_1)$ and mp lie on ml, and since $\Gamma(f^m/x_1) \notin \operatorname{conv}(J^{mt})$ and $mp \in \operatorname{conv}(J^{mt})$, we have $\|mv\| = \|mp - \Gamma(f^m)\| \leq \|\Gamma(f^m) - \Gamma(f^m/x_1)\| = 1$ for any m, a contradiction.

Case 2: Now assume that $Z \subset G$ for some noncompact face $G \in \mathcal{F}$ and that $\{a_{j_1}, \ldots, a_{j_t}\} \not\subset F$ for any compact face $F \in \mathcal{F}$. We prove that $f^m \notin G(\overline{J^{mt}})$ for some $m = m_0$ by induction on $\dim G$. If $\dim G = 1$, then $f \notin G(\overline{J^t})$, because it follows from Lemma 3.1 that the only point on tG which corresponds to a generator of $\overline{J^t}$, is an extremal point of $\operatorname{conv}(J^t)$ and certainly $a = a_{j_1} + \cdots + a_{j_t}$ is not an extremal point of $\operatorname{conv}(J^t)$, see Corollary 2.2. Now let $\dim G = p > 1$. We may assume that $\{a_{j_1}, \ldots, a_{j_t}\} \not\subset G'$ for any proper face G' of G. As if $\{a_{j_1}, \ldots, a_{j_t}\} \subset G'$ for some proper face G' of G, then G' is a noncompact face of G with $\dim G' < \dim G$ and we are through by induction.

Let $S := \{v \in \mathbb{R}^n \mid \langle v, u\rangle = c\}$ be the supporting hyperplane of $\operatorname{conv}(J)$ such that $S \cap \operatorname{conv}(J) = G$. Since G is a noncompact face, there exists j such that $u(j) = 0$. Consider $a_\lambda := a_{j_1} + \cdots + a_{j_t} - \lambda(0, \ldots, 1, \ldots, 0)$, 1 being at jth place, $\lambda \geq 0$. Notice that there exists $\lambda_0 > 0$ such that $a_{\lambda_0} \notin \operatorname{conv}(J)$. Let l_0 be the line segment joining a and a_{λ_0}. As $a \in l_0 \cap tG$, the intersection of l_0 with tG is a nonempty convex set. Let $l = l_0 \cap tG$ be the line segment joining a and $a_{\lambda'}$ where $a_{\lambda'}$ lies on some proper face tG' of tG and $\lambda' > 0$, as $\dim G' < \dim G$. Also $a_{\lambda'} < a$, so we have $a = a_{\lambda'} + w$, with $\|w\| = \lambda' > 0$. For any positive integer m, $ma_{\lambda'} \in mtG'$ and $\|ma - ma_{\lambda'}\| = m\|a - a_{\lambda'}\| = m\|w\| > 0$. Let for $m = m_0$, $m\|w\| \geq 1$. Then for $m = m_0$, ma and $ma - (0, \ldots, 1, \ldots, 0)$ lies on mtG, 1 being at j th place, so that $\Gamma(f^m/x_j) \in mtG$ which implies $f^m/x_j \in \overline{J^{mt}}$ and hence $f^m \notin G(\overline{J^{mt}})$ for $m = m_0$. $\square$

Let $S = K[x_1, \ldots, x_n, y_1, \ldots, y_r]$ be a bigraded polynomial ring with $\deg x_i = (1, 0)$ and $\deg y_j = (d_j, 1)$. Recall $J = (f_1, \ldots, f_r)$ where $f_j = x^{a_j}$ and $\deg f_j = d_j$. Let φ be the surjective homomorphism from S to $\mathcal{R}(J) = K[x_1, \ldots, x_n, f_1t, \ldots, f_rt]$, given by $x_i \mapsto x_i$ and $y_j \mapsto f_jt$ so that $S/L \cong \mathcal{R}(J)$ where the ideal L is generated by binomials of the type $g_1h_1 - g_2h_2$ where g_1, g_2 are monomials in x_i and h_1, h_2 are monomials in y_j. Notice that $\deg h_1 = \deg h_2$.

Now consider the fiber ring $\mathcal{F}(J) = \mathcal{R}(J)/\mathfrak{m}\mathcal{R}(J)$ of the ideal J where $\mathfrak{m} = (x_1, \ldots, x_n) \subset A$. Then $\mathcal{F}(J) \cong S/(L, \mathfrak{m}) \cong T/D$ and hence $\mathcal{F}(J)_{\mathrm{red}} \cong T/\operatorname{Rad} D$ where D is the image of the ideal L in $T = S/\mathfrak{m}$, and $T = K[y_1, \ldots, y_r]$. Let $\psi = \varphi \otimes S/\mathfrak{m} \colon T \to \mathcal{F}(J)$ be the induced epimorphism. We have $D = \operatorname{Ker} \psi$. Notice that the ideal D is generated by monomials and homogeneous binomials in the y_j. In fact, if $g_1h_1 - g_2h_2$ is a generator of L, then its image in T is a monomial, if one of the g_i belongs to $\mathfrak{m}$, and otherwise it is a homogeneous binomial. We have the following lemma:

Lemma 4.4. *Let $b = b_1 - b_2 \in D$ be a homogeneous binomial generator of D with $b_1 = y_{i_1}^{l_1} \cdots y_{i_u}^{l_u}$, $b_2 = y_{j_1}^{m_1} \cdots y_{j_v}^{m_v}$ and $\sum_{i=1}^{u} l_i = \sum_{j=1}^{v} m_j = t$. If the set $\{a_{i_1}, \ldots, a_{i_u}\} \subset G$ for some $G \in \mathcal{F}_c$, then also the set $\{a_{j_1}, \ldots, a_{j_v}\} \subset G$.*

Proof. As $b \in D$, we have $\psi(b) = 0$, i.e. $\psi(b_1) = \psi(b_2)$. Therefore we have $x^{l_1 a_{i_1}} \cdots x^{l_u a_{i_u}} = x^{m_1 a_{j_1}} \cdots x^{m_v a_{j_v}}$, and so $\sum_{p=1}^{u} l_p a_{i_p} = \sum_{k=1}^{v} m_k a_{j_k}$. Let the set $\{a_{i_1}, \ldots, a_{i_u}\} \subset G$ for some $G \in \mathcal{F}_c$. We show that $\{a_{j_1}, \ldots, a_{j_v}\} \subset G$. Let $S := \{v \in \mathbb{R}^n \mid \langle v, u \rangle = c\}$, be the supporting hyperplane of $\operatorname{conv}(J)$ such that $S \cap \operatorname{conv}(J) = G$.

We have $\langle \sum_{k=1}^{v} m_k a_{j_k}, u \rangle = \langle \sum_{p=1}^{u} l_p a_{i_p}, u \rangle = tc$. Suppose $\{a_{j_1}, \ldots, a_{j_v}\} \not\subset G$, then there exists at least one $k_0 \in \{1, \ldots, v\}$ such that $a_{j_{k_0}} \notin G$. Since $\langle a_{j_k}, u \rangle \geq c$ for all k, it follows that $\langle a_{j_{k_0}}, u \rangle > c$ which in turn implies that $\langle \sum_{k=1}^{v} l_k a_{j_k}, u \rangle > tc$, a contradiction. $\square$

We denote by $\mathcal{F}_c$ the set of compact faces, and by $\mathcal{F}_{mc}$ the set of maximal compact faces of $\operatorname{conv}(J)$. Let $F \in \mathcal{F}_{mc}$; we set $P_F = (y_j : a_j \notin F)$ and we denote by B_F the kernel of $\theta_F : K[y_j : a_j \in F] \to K[F] := K[f_j t : a_j \in F]$ where $\theta_F(y_j) = f_j t$.

With the notation introduced we have

Proposition 4.5. $\operatorname{Rad} D = (\bigcap_{F \in \mathcal{F}_{mc}} P_F, \sum_{F \in \mathcal{F}_{mc}} B_F) = \bigcap_{F \in \mathcal{F}_{mc}} (P_F, B_F)$.

Proof. For the proof we proceed in several steps.

1. Step: Let f be a monomial in T. We claim that $f \in \operatorname{Rad} D \iff f \in \bigcap_{F \in \mathcal{F}_{mc}} P_F$.

We may assume that f is squarefree. So let $f = y_{j_1} \ldots y_{j_k}$ with $j_1 < j_2 < \cdots < j_k$ and assume that $f \in \operatorname{Rad} D$. Then $f^{n_0} \in D$ for some integer n_0, and hence $\psi(f^{n_0}) = 0$. This implies that $x^{n_0 a_{j1}} \cdots x^{n_0 a_{jk}} \in \mathfrak{m} J^{n_0 k}$. Hence $x^{n a_{j1}} \cdots x^{n a_{jk}}$ is not a minimal generator of J^{nk} for any $n \geq n_0$. Now Theorem 4.3 implies that $\{a_{j1}, \ldots, a_{jk}\} \not\subset F$ for any compact face $F \in \mathcal{F}$. This shows that $f \in \bigcap_{F \in \mathcal{F}_{mc}} P_F$.

Conversely, assume that $f \in \bigcap_{F \in \mathcal{F}_{mc}} P_F$. Then $\{a_{j_1}, \ldots, a_{j_k}\} \not\subset F$ for any $F \in \mathcal{F}_{mc}$. This implies that $\{a_{j_1}, \ldots, a_{j_k}\} \not\subset F$ for any compact face. From Theorem 4.3 we conclude that there exists an integer m such that $(x^{a_{j_1}} \cdots x^{a_{j_k}})^m \in \mathfrak{m} J^{km}$. Since $\psi(f^m) = (x^{a_{j_1}} \cdots x^{a_{j_k}})^m$ it follows that $f^m \in D$, and hence $f \in \operatorname{Rad} D$.

2. Step: $D \subset (\bigcap_{F \in \mathcal{F}_{mc}} P_F, \sum_{F \in \mathcal{F}_{mc}} B_F)$.

It follows from the first step that all monomial generators in D belong to the ideal $(\bigcap_{F \in \mathcal{F}_{mc}} P_F, \sum_{F \in \mathcal{F}_{mc}} B_F)$. Now let $b = b_1 - b_2$ be one of the homogeneous binomial generators of D with $b_1 = y_{i_1}^{l_1} \cdots y_{i_u}^{l_u}$, $b_2 = y_{j_1}^{m_1} \cdots y_{j_v}^{m_v}$ and $\sum_{i=1}^{u} l_i = \sum_{j=1}^{v} m_j = t$. As $b \in D$, we have $\psi(b) = 0$, i.e. $\psi(b_1) = \psi(b_2)$. Therefore we have $x^{l_1 a_{i_1}} \cdots x^{l_u a_{i_u}} = x^{m_1 a_{j_1}} \cdots x^{m_v a_{j_v}}$, and so $\sum_{p=1}^{u} l_p a_{i_p} = \sum_{k=1}^{v} m_k a_{j_k}$. We show that $b \in \sum_{F \in \mathcal{F}_{mc}} B_F$, if $b \notin \bigcap_{F \in \mathcal{F}_{mc}} P_F$. In fact, if $b \notin \bigcap_{F \in \mathcal{F}_{mc}} P_F$, then one of the b_i, say $b_1 \notin \bigcap_{F \in \mathcal{F}_{mc}} P_F$. This implies that $\{a_{11}, \ldots, a_{1u}\} \in G$ for some compact face $G \in \mathcal{F}_{mc}$ and then from Lemma 4.4, $\{a_{21}, \ldots, a_{2v}\} \in G$. Hence, $b = b_1 - b_2 \in B_G$.

3. Step: $\sum_{F \in \mathcal{F}_{mc}} B_F \subset D$.

Notice that $B_F = \operatorname{Ker} \theta_F$ and $D = \operatorname{Ker} \psi$. Certainly, for each $F \in \mathcal{F}_{mc}$, $\operatorname{Ker} \theta_F \subset \operatorname{Ker} \psi$ and hence $\sum_{F \in \mathcal{F}_{mc}} B_F \subset D$.

4. Step: $\bigcap_{F\in\mathcal{F}_{mc}}(P_F, B_F) = (\bigcap_{F\in\mathcal{F}_{mc}} P_F, \sum_{F\in\mathcal{F}_{mc}} B_F)$.

For each $F \in \mathcal{F}_{mc}$, let $Q_F = (P_F, B_F)$, and let $M = \bigcap_{F\in\mathcal{F}_{mc}} P_F$ and $B = \sum_{F\in\mathcal{F}_{mc}} B_F$. In order to show that $(M, B) = \bigcap_{F\in\mathcal{F}_{mc}} Q_F$, we proceed in the following steps:

(i) First we show $(M, B) \subset \bigcap_{F\in\mathcal{F}_{mc}} Q_F$. Clearly, for each $F \in \mathcal{F}_{mc}$, $M \subset Q_F$. Now we also prove that $B \subset Q_F$ for all $F \in \mathcal{F}_{mc}$. Take $b = b_1 - b_2 \in B$ with $b_1 = y_{i_1}^{l_1} \cdots y_{i_u}^{l_u}$, $b_2 = y_{j_1}^{m_1} \cdots y_{j_v}^{m_v}$ and $\sum_{i=1}^{u} l_i = \sum_{j=1}^{v} m_j = t$. Suppose that $b \notin B_G$, then we prove $b \in P_G$. As $b \notin B_G$, it implies that for one of the b_i, say for b_1, there exists $y_{i_p}|b_1$ such that $a_{i_p} \notin G$. Once we show that there exists also some $k \in \{1, \dots, v\}$ such that $y_{j_k}|b_2$ and $a_{j_k} \notin G$, then it will imply that $b_1, b_2 \in P_G$ and hence $b \in P_G$. Suppose this is not the case, then $\{a_{j_1}, \dots, a_{j_v}\} \in G$. But then from Lemma 4.4, we have $\{a_{i_1}, \dots, a_{i_v}\} \in G$ which is a contradiction. Hence we have$(M, B) \subset \bigcap_{F\in\mathcal{F}_{mc}} Q_F$.

(ii) Notice that for each $F \in \mathcal{F}_{mc}$, Q_F is a prime ideal. Indeed, Q_F being the kernel of the surjective map $\pi_F : K[y_1, \dots, y_r] \to K[f_i t : a_i \in F]$ given by $\pi_F(y_j) = f_j t$, if $a_j \in F$ and $\pi_F(y_j) = 0$, if $a_j \notin F$, the assertion follows.

(iii) We claim that $\{Q_F : F \in \mathcal{F}_{mc}\}$ is the set of all the minimal prime ideals containing (M, B). Let P be any prime ideal containing (M, B), then it implies that $P \supset M = \bigcap_{\mathcal{F}_{mc}} P_F$ and so $P \supset P_G$ for some $G \in \mathcal{F}_{mc}$. Also, $P \supset B = \sum B_F$. Hence $P \supset Q_G$.

(iv) We claim (M, B) is a radical ideal, that is, $\mathrm{Rad}(M, B) = (M, B)$. This amounts to prove that for all Q_F, $(M, B)T_{Q_F} = Q_F T_{Q_F}$. Fix $G \in \mathcal{F}_{mc}$, the set $\{y_i : a_i \in G\} \subset T \backslash Q_G$ and hence all y_i such that $a_i \in G$ are invertible in T_{Q_G}. For all P_F, $F \neq G$, there exists at least one $y_j \in P_F$ such that $y_j \in G$, as otherwise $P_F \subset P_G$ which implies $F \supset G$, a contradiction. Hence for all $F \neq G$, $P_F T_{Q_G} = T_{Q_G}$. Therefore we have $(M, B)T_{Q_G} = (\bigcap_{F\in\mathcal{F}_{mc}} P_F, \sum_{F\in\mathcal{F}_{mc}} B_F)T_{Q_G} = (P_G, \sum_{F\in\mathcal{F}_{mc}} B_F)T_{Q_G} = (P_G, B_G)T_{Q_G} = Q_G T_{Q_G}$.

Since by (iii) we have $\mathrm{Rad}(M, B) = \bigcap_{F\in\mathcal{F}_{mc}} Q_F$ it follows then that $(M, B) = \bigcap_{F\in\mathcal{F}_{mc}} Q_F$. Now by Step 1, Step 2 and Step 3, one has

$$D \subset (\bigcap_{F\in\mathcal{F}_{mc}} P_F, \sum_{F\in\mathcal{F}_{mc}} B_F) \subset \mathrm{Rad}\, D.$$

Finally by Step 4, we have $(\bigcap_{F\in\mathcal{F}_{mc}} P_F, \sum_{F\in\mathcal{F}_{mc}} B_F) = \bigcap_{F\in\mathcal{F}_{mc}}(P_F, B_F)$ which is a radical ideal. Hence we have $\mathrm{Rad}\, D = (\bigcap_{F\in\mathcal{F}_{mc}} P_F, \sum_{F\in\mathcal{F}_{mc}} B_F) = \bigcap_{F\in\mathcal{F}_{mc}}(P_F, B_F)$. $\square$

We denote by $\mathrm{Min}(R)$ the set of minimal prime ideals of a ring R.

Corollary 4.6. *Let $I \subset A$ be a monomial ideal. Then there is an injective map*

$$\mathcal{F}_{mc} \to \mathrm{Min}(\mathcal{F}(I)).$$

This map is bijective if I is an extremal ideal.

Proof. Let J be the minimal monomial reduction ideal of I. Then J is an extremal ideal. From above proposition we have $\mathcal{F}(J)_{\mathrm{red}} \cong T/\bigcap_{F\in\mathcal{F}_{mc}}(P_F, B_F)$ where

(P_F, B_F) is a prime ideal for each $F \in \mathcal{F}_{mc}$. Hence there is a bijective map

$$\rho_1 \colon \mathcal{F}_{mc} \to \mathrm{Min}(\mathcal{F}(J))$$

given by $F \mapsto (P_F, B_F)/D$.

As $\mathcal{F}(I)$ is integral over $\mathcal{F}(J)$, for each $P \in \mathrm{Min}(\mathcal{F}(J))$ there exists a minimal prime ideal $Q \in \mathrm{Min}(\mathcal{F}(I))$ such that $P = Q \cap \mathcal{F}(J)$. Therefore there exists an injective map ρ_2 from $\mathrm{Min}(\mathcal{F}(J))$ to $\mathrm{Min}(\mathcal{F}(I))$, and hence $\rho = \rho_2 \circ \rho_1 \colon \mathcal{F}_{mc} \to \mathrm{Min}(\mathcal{F}(I))$ is the desired injective map. Finally, if I is extremal, then $I = J$ and $\rho = \rho_1$ is a bijection. □

Next corollary gives us a combinatorial characterization of the fiber ring of an extremal ideal J to be a domain.

Corollary 4.7. *Let $J = (x^{a_1}, \ldots, x^{a_r})$ be an extremal ideal. Then the following conditions are equivalent:*

(1) *The fiber ring $\mathcal{F}(J)$ is a domain;*
(2) *The reduced fiber ring $\mathcal{F}(J)_{\mathrm{red}}$ is a domain;*
(3) $|\mathcal{F}_{mc}| = 1$.

Proof. (1) $\Rightarrow$ (2) is obvious, and (2) $\iff$ (3) follows from Corollary 4.6.
(3) $\Rightarrow$ (1): Let $|\mathcal{F}_{mc}| = 1$. Then it follows from Proposition 4.5 that $\mathrm{Rad}\, D = (B_F, P_F)$ where $F \in \mathcal{F}_{mc}$. Notice that as there is only one maximal compact face F, the ideal P_F is the zero ideal. Hence $(P_F, B_F) = B_F$. Also by Step 3 in the proof of Proposition 4.5 we have $B_F \subset D$. Therefore we have $\mathrm{Rad}\, D = D = B_F$ which is a prime ideal. Hence $\mathcal{F}(J) \cong T/D$ is a domain. □

By the above corollary the fiber ring of an extremal ideal J is a domain if and only if there is only one maximal compact faces of $\mathrm{conv}(J)$. But in general the property of being reduced cannot be characterized in terms of combinatorial properties of $\mathrm{conv}(J)$, as the the following simple example demonstrates:

Example 4.8. Consider the two extremal ideals $J_1 = (x^6, x^2y, xy^2, y^6)$ and $J_2 = (x^8, x^6y, x^2y^7, y^{12})$ in the polynomial ring $A = K[x, y]$. It is easy to see that $\mathrm{conv}(J_1)$ and $\mathrm{conv}(J_2)$ have the same face lattices. Nevertheless the fiber ring of the ideal J_1 given by $\mathcal{F}(J_1) \cong K[y_1, y_2, y_3, y_4]/(y_1y_4, y_2y_4, y_1y_3)$ is reduced while the fiber ring of the ideal J_2 given by $\mathcal{F}(J_2) \cong K[y_1, y_2, y_3, y_4]/(y_1y_4, y_2y_4^2, y_2^2y_4 - y_1y_3^2, y_1^2y_3)$ is not reduced.

Next we define an inverse system of semigroup rings $K[F]$ for $F \in \mathcal{F}_c$(set of compact faces of $\mathrm{conv}(I)$) where $K[F] = K[f_it \colon a_i \in F]$ with $f_i = x^{a_i}$. For $G \subset F$, define the ring homomorphism $\pi_{GF} : K[F] \to K[G]$, given by $\pi_{GF}(f_it) = f_it$, if $a_i \in G$ and $\pi_{GF}(f_it) = 0$, otherwise. Notice that π_{GF} is well defined. To see this, we need to show that if $f_{i_1}f_{i_2}\cdots f_{i_k}t^k = f_{j_1}f_{j_2}\cdots f_{j_k}t^k$ where $\{a_{i_1}, \ldots, a_{i_k}\}, \{a_{j_1}, \ldots, a_{j_k}\} \subset F$, then $\pi_{GF}(f_{i_1}f_{i_2}\cdots f_{i_k}t^k) = \pi_{GF}(f_{j_1}f_{j_2}\cdots f_{j_k}t^k)$. If $\pi_{GF}(f_{i_1}\cdots f_{i_k}t^k) = 0$, then $\{a_{i_1}, \ldots, a_{i_k}\} \not\subset G$. Since $y_{i_1}\cdots y_{i_k} - y_{j_1}\cdots y_{j_k} \in D$ it follows from Lemma 4.4 that $\{a_{j_1}, \ldots, a_{j_k}\} \not\subset G$, too. Hence $\pi_{GF}(f_{j_1}\cdots f_{j_k}t^k) = 0$. On the other hand, if $\pi_{GF}(f_{i_1}\cdots f_{i_k}t^k) \neq 0$, then $\pi_{GF}(f_{j_1}\cdots f_{j_k}t^k) \neq 0$, and so

$$\pi_{GF}(f_{i_1}\cdots f_{i_k}t^k) = f_{i_1}\cdots f_{i_k}t^k = f_{j_1}\cdots f_{i_k}t^k = \pi_{GF}(f_{j_1}\cdots f_{i_k}t^k).$$

Hence $\pi_{GF}(f_{i_1}\cdots f_{i_k}t^k)=\pi_{GF}(f_{j_1}\cdots f_{j_k}t^k)$ in both cases.

Also we may notice that for $H\subset G\subset F$ and $F\in\mathcal{F}_c$, one has $\pi_{HG}\circ\pi_{GF}=\pi_{HF}$. Hence the inverse system is well defined.

Theorem 4.9. $F(J)_{\mathrm{red}}\cong \varprojlim_{F\in\mathcal{F}_c}K[F]$.

Proof. For each $F\in\mathcal{F}_c$ consider the ring homomorphism π_F from $K[y_1,\ldots,y_r]$ to $K[F]$ given by $\pi_F(y_j)=f_jt$, if $a_j\in F$ and $\pi_F(y_j)=0$, if $a_j\notin F$.

Notice that $\operatorname{Ker}\pi_F$ is equal to the ideal $Q_F:=(B_F,P_F)$. We define the map

$$\pi: K[y_1,\ldots y_r]\longrightarrow \bigoplus_{F\in\mathcal{F}_c}K[F],$$

given by $\pi=(\pi_F)_{F\in\mathcal{F}_c}$. We have $\operatorname{Ker}\pi=\bigcap_{F\in\mathcal{F}_c}Q_F=\bigcap_{F\in\mathcal{F}_c}(B_F,P_F)$. We claim that for all $G\subset F$ one has $Q_F\subset Q_G$. Indeed, for all $G\subset F$, $P_F\subset P_G$ and by the proof of Proposition 4.5, Step 4(i), we have $B_F\subset(B_G,P_G)$. It follows that

$$\operatorname{Ker}\pi=\bigcap_{F\in\mathcal{F}_{mc}}Q_F.$$

Therefore Proposition 4.5 implies that $\operatorname{Ker}\pi=\operatorname{Rad}D$. Thus we have

$$K[y_1,\ldots,y_r]/\operatorname{Ker}\pi\cong F(J)_{\mathrm{red}}.$$

It remains to show that $\operatorname{Im}(\pi)=\varprojlim_{F\in\mathcal{F}_c}K[F]$. First notice that $\operatorname{Im}(\pi)\subset \varprojlim_{F\in\mathcal{F}_c}K[F]$, since $\pi_{GF}\circ\pi_F=\pi_G$ for all $G\subset F$.

Now let $v=(m_F)_{F\in\mathcal{F}_c}\in\varprojlim_{F\in\mathcal{F}_c}K[F]$. We may assume that for each $F\in\mathcal{F}_c$, the element m_F is a monomial in $K[F]$ since all homomorphisms in the inverse system are multigraded. For each $F\in\mathcal{F}_c$, we choose $g_F\in K[y_1,\ldots,y_r]$ such that $\pi_F(g_F)=m_F$ and with the property that whenever $m_F=m_G$ in $K[x_1,\ldots,x_n,t]$ then it implies $g_F=g_G$. (Notice that for each $F\in\mathcal{F}$, the K-algebra K[F] can be naturally embedded in the K-algebra $K[x_1,\ldots,x_n,t]$).

Let $Z=\{m_F: m_F\neq 0, F\in\mathcal{F}_c\}=\{m_1,\ldots,m_l\}$. For each $i=1,\ldots,l$, we define the set $A_i=\{F\in\mathcal{F}_c: m_F=m_i\}$. We claim that for each A_i one has $\bigcap_{F\in A_i}F\in A_i$. Fix an i, and notice that it is enough to show that for any $F,G\in A_i$ we have $F\cap G\in A_i$. Let $m_F=f_{i_1}\cdots f_{i_p}t^p=f_{j_1}\cdots f_{j_p}t^p=m_G$. Then it follows by Lemma 4.4 that the sets $\{a_{i_1},\ldots,a_{i_p}\},\{a_{j_1},\ldots,a_{j_p}\}\subset F\cap G=H$. Therefore $\pi_{HF}(m_F)=m_F$ and $\pi_{HG}(m_G)=m_G$. Also as $v=(m_F)_{F\in\mathcal{F}_c}\in\varprojlim_{F\in\mathcal{F}_c}K[F]$ we have $\pi_{HF}(m_F)=m_H=\pi_{HG}(m_G)$. Hence $m_G=m_F=m_H$, so $H\in A_i$. Hence $H_i=\bigcap_{F\in A_i}F\in A_i$, $i=1,\ldots,l$.

For each i, we choose a monomial $g_{H_i}\in K[y_1,\ldots,y_r]$ such that $\pi_{H_i}(g_{H_i})=m_{H_i}$. For all $F\in A_i$, we define $g_F=g_{H_i}$, $i=1,\ldots,l$ and for all $F\in\mathcal{F}_c\setminus\bigcup_{i=1}^l A_i$, we define $g_F=0$. Notice that for all $F\in\mathcal{F}_c$, we have $\pi_F(g_F)=m_F$. Indeed, let $F\in\mathcal{F}_c$. If $F\in\mathcal{F}_c\setminus\bigcup_{i=1}^l A_i$, then $g_F=0=m_F$ and we have $\pi_F(g_F)=m_F$. If $F\in A_i$ for some i, then as we have $\pi_{H_iF}\circ\pi_F=\pi_{H_i}$ and $\pi_{H_i}(g_F)=m_{H_i}=m_F$, it follows by the very definition of the map π_{H_iF} that $\pi_F(g_F)=m_F$. Moreover, by our choice of the g_F we also have $g_F=g_G$ whenever $m_F=m_G$.

Now let $S=\{g_F: F\in\mathcal{F}_{mc}\}$, and let $g=\sum_{g_F\in S}g_F$. We claim that $\pi(g)=v$, i.e. $\pi_G(g)=m_G$ for all $G\in\mathcal{F}_c$. Notice that it is enough to show that $\pi_G(g)=m_G$ for

all $G \in \mathcal{F}_{mc}$. In fact, if $H \in \mathcal{F}_c$ there exists $G \in \mathcal{F}_{mc}$ such that $H \subset G$, and since $\pi_G(g) = m_G$, we have $\pi_H(g) = \pi_{HG}(\pi_G(g)) = \pi_{HG}(m_G) = m_H$.

Now let $G \in \mathcal{F}_{mc}$. We claim that $\pi_G(g_F) = 0$ for all $g_F \neq g_G$, so that we have $\pi_G(g) = m_G$, as asserted.

To prove this claim, let $g_F = y_{i_1} \cdots y_{i_p}$ and suppose that $\pi_G(g_F) \neq 0$. Then we have $\{a_{i_1}, \ldots, a_{i_p}\} \subset G \cap F$. Let $H = G \cap F$, then $H \in \mathcal{F}_c$. Since $v \in \varprojlim_{F \in \mathcal{F}_c} K[F]$ and H is a common face of F and G, we have $\pi_{HF}(m_F) = m_H = \pi_{HG}(m_G)$. As $\{a_{i_1}, \ldots, a_{i_p}\} \subset H$, we have $0 \neq m_F = \pi_{HF}(m_F) = m_H = \pi_{HG}(m_G) = m_G$. Hence $g_F = g_G$, a contradiction. $\square$

The analytic spread ℓ of any ideal I in a Noetherian local ring $(R, \mathfrak{m})$ is given by the Krull dimension of the fiber ring $\mathcal{F}(I)$ of I. It has been shown by Carles Bivia-Ausina [4] that the analytic spread of any non-degenerate ideal $I \subset \mathbb{C}[[x_1, \ldots, x_n]]$ is equal to the $c(I) + 1$ where

$$c(I) = \max\{\dim F : F \text{ is a compact face of} \operatorname{conv}(I)\}.$$

Next we show that for monomial ideals this result is an immediate consequence of our structure theorem (Theorem 4.9).

Corollary 4.10. *Let $I \subset A = K[x_1, \ldots, x_n]$ be any monomial ideal. Let $\ell = \dim \mathcal{F}(I)$ be the analytic spread of ideal I. Then*

$$\ell = c(I) + 1 = \max\{\dim F : F \textit{ is a compact face of } \operatorname{conv}(I)\} + 1.$$

Proof. Let J be the minimal monomial reduction ideal of I. We have $\ell = \dim \mathcal{F}(I) = \dim \mathcal{F}(J) = \dim \mathcal{F}(J)_{\text{red}}$. By Theorem 4.9, we have $\mathcal{F}(J)_{\text{red}} = \varprojlim_{F \in \mathcal{F}_c} K[F] \subset \bigoplus_{F \in \mathcal{F}_c} K[F]$. Therefore $\dim(\mathcal{F}(J)) \leq \max\{\dim K[F] : F \in \mathcal{F}_c\}$. As $\dim K[F] = \dim F + 1$, it follows that $\ell \leq c(I) + 1$.
For proving $\ell \geq c(I) + 1$, we notice that the canonical homomorphisms

$$\bar{\pi}_G \colon \varprojlim_{F \in \mathcal{F}_c} K[F] \to K[G]$$

are surjective for all $G \in \mathcal{F}_c$. Indeed, if m is a monomial in $K[G]$ and $v = (m_F)_{F \in \mathcal{F}_c} \in \varprojlim_{F \in \mathcal{F}_c} K[F]$ with

$$m_F = \begin{cases} m, & \text{if } \operatorname{supp}(m) \subset F, \\ 0, & \text{if } \operatorname{supp}(m) \subset F, \end{cases}$$

then $\bar{\pi}_F(v) = m$. Here $\operatorname{supp}(m)$ of some monomial $m = x_1^{a_1} \cdots x_n^{a_n} \in A$ is defined to be $\operatorname{supp}(m) = \{a_i : a_i \neq 0\}$.
It follows that $\dim F(J) \geq \dim K[F]$ for all $F \in \mathcal{F}_{mc}$. Therefore we have $\ell \geq c(I)+1$, as desired. $\square$

5. On the reduction number of a monomial ideal

In this section we consider the reduction number of a monomial ideal $I \in A$ with respect to the minimal monomial reduction ideal J. We show in Corollary 5.3 that if I^m is integrally closed for $m \leq \ell$ then I is normal and the reduction number of I with respect to J is less than $\ell - 1$. Here ℓ denotes the analytic spread of the monomial ideal I and the reduction number of an ideal I with respect to J is defined to be the minimum of m such that $JI^m = I^{m+1}$.

Theorem 5.1. *Let $I \subset A = K[x_1, \ldots, x_n]$ be a monomial ideal and J its minimal monomial reduction ideal. Let ℓ be the analytic spread of I. Then*
$$\overline{I^m} = J\overline{I^{m-1}} \quad \textit{for all} \quad m \geq \ell.$$

Proof. We may assume I is a proper ideal, and let $I = (x^{a_1}, x^{a_2}, \ldots, x^{a_s})$ where $f_i = x^{a_i} = x_1^{a_i(1)} x_2^{a_i(2)} \cdots x_n^{a_i(n)}$ for $i = 1, \ldots, s$. Without loss of generality, let $J = (x^{a_1}, x^{a_2}, \ldots, x^{a_r})$ be the minimal monomial reduction ideal of I so that $\operatorname{ext}(I) = \{a_1, \ldots, a_r\}$. Let $m \geq \ell$, we show $\overline{I^m} \subset J\overline{I^{m-1}}$, the other inclusion being trivial. Let $x^b \in \overline{I^m} = \overline{J^m}$ where $x^b = x_1^{b(1)} \cdots x_n^{b(n)}$.

For the proof we consider the following two cases:

Case 1. $b \in F$ where F is a face of $\operatorname{conv}(I^m)$.

First we claim that $b = b_1 + v$ where $b_1 \in G$ for some compact face G of $\operatorname{conv}(I^m)$ and $v \in \mathbb{R}^n_{\geq 0}$. If F is a compact face, then we take $v = 0$ and $b_1 = b$. Now let F be a noncompact face. We prove the claim by induction on $\dim F$. If $\dim F = 1$, then clearly $b = ma_i + v$ where $v \in \mathbb{R}^n_{\geq 0}$ for some $a_i \in \operatorname{ext}(I)$. Now let $\dim F = t > 1$. Let $S = \{v \in \mathbb{R}^n \mid \langle v, u \rangle = c\}$ (where $u = (u(1), \ldots, u(n)) \in \mathbb{R}^n$, $c \in \mathbb{R}$) be a supporting hyperplane of $\operatorname{conv}(I^m)$ such that $S \cap \operatorname{conv}(I^m) = F$. Since F is an noncompact face there exists $u(j)$ such that $u(j) = 0$. Consider $b_\lambda := b - \lambda(0, \ldots, 1, \ldots, 0)$, 1 being at jth place, $\lambda \geq 0$. Notice that there exists $\lambda_0 > 0$ such that $b_{\lambda_0} \notin \operatorname{conv}(I^m)$. Let l_0 be the line segment joining b and b_{λ_0}. The intersection of l_0 with F, is nonempty and therefore is a convex set. It follows that $l = l_0 \cap F$ is a line segment joining b and $b_{\lambda'}$ where $b_{\lambda'}$ lies on some proper face F' of F and $\lambda' \geq 0$. Therefore $bb_{\lambda'} + w$ with $b_{\lambda'} \in F'$ and $w \in \mathbb{R}^n_{\geq 0}$. By induction, $b_{\lambda'} = b_1 + w'$ where $b_1 \in G$ for some compact face G and $w' \in \mathbb{R}^n_{\geq 0}$. Hence $b = b_1 + v$ with $v = w + w' \in \mathbb{R}^n_{\geq 0}$. Hence the claim.

As G is a compact face we have $\dim G \leq \ell$ by Corollary 4.10. Now since $b_1 \in G$ and there exists $p \leq \ell$ affinely independent vectors $\{a_{i_1}, \ldots, a_{i_p}\} \subset \operatorname{ext}(I)$ such that $b_1 = \sum_{j=1}^p k_j a_{i_j}$ with $\sum k_i = m$. Since $p \leq \ell \leq m$, there exists $a_{i_{j_0}}$ such that $b_1 - a_{i_{j_0}} \in \operatorname{conv}(I^{m-1})$. Therefore, $b - a_{i_{j_0}} = b_1 - a_{i_{j_0}} + v \in \operatorname{conv}(I^{m-1}) \cap \mathbb{N}^n = \Gamma(\overline{I^{m-1}})$. Hence $b \in \Gamma(J\overline{I^{m-1}})$.

Case 2. $b \notin F$ for any face F of $\operatorname{conv}(I^m)$.

Let $f = x^b$. We may assume that $f \in G(\overline{J^m})$. Without loss of generality, let $x_1 | f$. Since $f \in G(\overline{J^m})$, $g = f/x_1 \notin \overline{J^m}$. Hence $b \in \operatorname{conv}(I^m)$ and $\Gamma(g) \notin \operatorname{conv}(I^m)$. Let l be the line segment joining b and $\Gamma(g)$. Then l intersects $\operatorname{conv}(I^m)$ at some point $a \in F$ where F is a face of $\operatorname{conv}(I^m)$. Hence, $b = a + v$ where $v \in \mathbb{R}^n_{\geq 0}$. Now by the proof of first case, we may write $a = a_1 + v_1$ such that $a_1 \in G$ for some compact face G of $\operatorname{conv}(I^m)$ and $v_1 \in \mathbb{R}^n_{\geq 0}$. Hence $b = a_1 + w$ where $w = v + v_1 \in \mathbb{R}^n_{\geq 0}$. Hence as in the above case, we get that $x^b \in J\overline{I^{m-1}}$. $\square$

Remark 5.2. There is a related result by Wiebe. He shows that for the maximal graded ideal $\mathfrak{m}$ in a positive normal affine semigroup ring S of dimension d one has $\overline{\mathfrak{m}^{n+1}} = \mathfrak{m}\overline{\mathfrak{m}^n}$ for all $n \geq d-2$, and that $\overline{\mathfrak{a}^{n+1}} = \mathfrak{a}\overline{\mathfrak{a}^n}$ for all $n \geq d-1$ if $\mathfrak{a} \subset S$ is an integrally closed ideal, see [1, Theorem 2.1].

Corollary 5.3. *Let I^a be integrally closed for all $a \leq \ell - 1$, then $I^\ell = JI^{\ell-1}$ and I is normal, i.e. I^a is integrally closed for all a.*

Proof. By the above theorem we have $\overline{I^\ell} \subset J\overline{I^{\ell-1}}$, and since $\overline{I^{\ell-1}} = I^{\ell-1}$, we see that $\overline{I^\ell} \subset JI^{\ell-1}$. Hence $I^\ell = JI^{\ell-1}$.
Also, $\overline{I^\ell} = J\overline{I^{\ell-1}} = JI^{\ell-1} \subset I^\ell \subset \overline{I^\ell}$. Hence $\overline{I^\ell} = I^\ell$. By applying induction on a, one has $\overline{I^a} = I^a$ for all a. □

Remarks 5.4. (a) Corollary 5.3 is a generalization of a result by Reid, Roberts and Vitulli [11, Proposition 2.3] . They proved that if $I \subset A = K[x_1, \ldots, x_n]$ is a monomial ideal and I^m is integrally closed for $m \leq n - 1$, then I is a normal ideal.

(b) In Corollary 5.3, once we assume that the monomial ideal I is normal, then the bound on the reduction number with respect to monomial reductions can be obtained as a consequence of a theorem by Valabrega-Valla [14] and the improved version of the Briancon-Skoda theorem due to Aberbach and Huneke [2]. Infact, if I is a normal monomial ideal, then $R(I)$ is Cohen-Macaulay and hence $F(I)$ is Cohen-Macaulay. Thus by Valabrega-Valla [14] and Aberbach-Huneke [2], the reduction number of I with respect to monomial reductions is less than the analytic spread ℓ of I. I am thankful to Prof. Verma for this information.

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